BLUEBIRD MATH CIRCLE
Alliance of Indigenous Math Circles

Issue 35 Recap
Sequences of Subsets: Binary Gray Code

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Introduction

Donna Fernandez introduced our math circle. Before we tell you all about it, here are our tools and supplies:

- Newsletter: https://aimathcircles.org/issue-34-sequences-of-subsets-full-color-pdf/
- Jamboards (including participants' work):
  https://jamboard.google.com/d/13biZ6oJtqQtsGUC6KWfCWdXBnt8FRn2ojckzPQPdFVI/viewer
  https://jamboard.google.com/d/14rAnKgMXlZHpcy1ReP6AAC_DHpofVHcbuOHtlvgitjE/viewer
  https://jamboard.google.com/d/1c--maeOzU_qPCzvJ3zaaVNvRb9Lo7m67L66sp5mZG5w/viewer

Sequences of Subsets

We first reviewed the notions of an ordered triple. An ordered triple is a list of three numbers, given in a specific order. Each of the numbers is called a *coordinate* of the ordered triple. In the ordered triples we consider, each coordinate is either 0 or 1. The usual way to write an ordered triple uses commas and parentheses: (1,0,1). But because we are limiting our coordinates to 0 or 1, we will write this ordered triple as 101. We do NOT intend this to be an integer!

We found that there are eight ordered triples with coordinates 0 or 1:

000
001
010
011
100
101
110
111.

To examine more closely the structure of this set of ordered triples, we looked at the following ways to list them all:

List A	*List B*	*List C*	*List D*	*List E*
000	101	111	100	110
001	110	100	101	111
010	010	101	111	101
100	100	100	110	100
110	001	011	010	000
101	010	010	011	010
011	111	001	001	011
111	000	000	000	001

But in fact one of these lists is WRONG! If you look closely, list C contains one ordered pair twice, and is missing another ordered pair.

This exercise led us to various insights about this set of eight ordered pairs:

- There are two ordered pairs with just one kind of digit: 000, 111.
- There are three ordered pairs with one 1 and two 0s: 100, 010, 001.
- There are three ordered pairs with one 0 and two 1s: 110, 101, 011.

We then set ourselves the main task of the session. List D above has the property that passing from each entry to the next (reading down), we change only one coordinate. For example, going from the second to the third ordered triple (101 to 111) we change only the second coordinate: the other two coordinates remain the same.

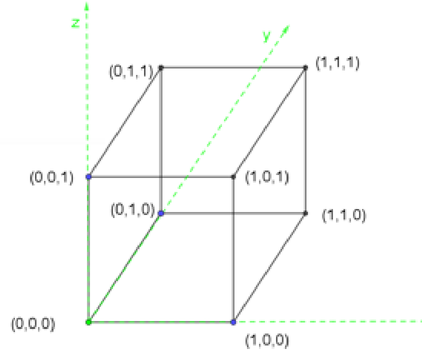
List A (which is a complete list of the eight ordered triples) does not have this property. For example, going from the second to the third ordered triple in list A (001 to 010) we must change both the second and the third coordinate.

A careful examination showed that list E has this property as well, but lists A and B lack this property. (And list C misses some ordered triples altogether.)

We then went into breakout rooms. Our task was to find some more lists like list D, for which only one coordinate changes each time. Some record of our work can be seen in the jamboards referenced above. Here are some results:

- There are many such lists.
- Often, we started a list, and got into a dead end: we had to change two coordinates. In that case we could backtrack.
- We started thinking of the possibilities visually. If we start at 000, to what other ordered triples can we move? We can move to 001, 010, or 100: three possibilities. So each ordered triple is connected to three other ordered triples. People started drawing various diagrams of this situation.
- Participants started talking about the ‘distance’ between two ordered triples. Some could be adjacent to each other in one of our lists. Some needed an ‘intermediate’ ordered triple, and some needed two intermediate ordered triples.

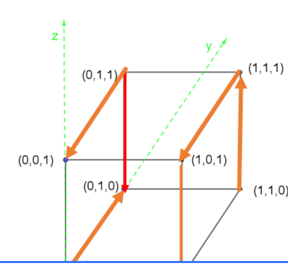
At this point the discussion connected to the problem from session 34. The term ‘coordinate’ suggested to some participants the structure of three-dimensional coordinate space. In this space, the eight points are the vertices of a cube:



We can visualize each of our lists as a path along the edges of the cube, visiting each vertex (each corner) exactly once. For example, here are the paths for lists D and E:

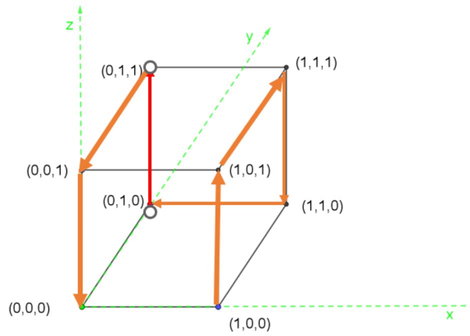
List E

110
111
101
100
000
010
011



LIST D

100
101
111
110
010
011
001
000



This geometric interpretation gave us a new insight into the idea of a ‘distance’ between two ordered triples. This distance can be interpreted as the number of edges we must follow to get from one to the other. It turns out that this is a very important concept in information theory, called the *Hamming Distance*.

Some participants connected this work with the ideas from Bluebird Session 34. And in fact, the two tasks (lists of subsets and lists of ordered triples) are *isomorphic*. The isomorphism is given by the notion of a characteristic function. This is a mapping from the set of subsets of {a,b,c} to the set of ordered triples of 0s and 1s:

Subset	**Is a an element?**	**Is b an element?**	**Is c an element?**	**The list**
	No	No	No	(0,0,0)
{a}	Yes	No	No	(1,0,0)
{b}	No	Yes	No	(0,1,0)
{c}	No	No	Yes	(0,0,1)
{a,b}	Yes	Yes	No	(1,1,0)
{a,c}	Yes	No	Yes	(1,0,1)
{b,c}	No	Yes	Yes	(0,1,1)
{a,b,c}	yes	yes	Yes	(1,1,1)

Using this mapping, the tasks of listing ordered triples and the task of sequencing the subsets of {a,b,c} are exactly the same.

We briefly talked about why this task is important. Computers often have to do something as simple as *count*. Numbers in a computer are coded as ‘strings’ of 1s and 0s. If we code the numbers from 0 to 7 using our list D, then the counting is very efficient. We need only ‘flip’ one digit at a time:

0	000
1	100
2	110
3	010
4	011
5	111
6	101
7	001

This way of coding the integers from 0 to 7 is called *Binary Gray Code*, and finds application in many areas of mathematics. Frank Gray (and not ‘Grey’) was the name of the person who first pointed out how useful this coding is.

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