We looked this week at some ceramics created by artists of the Hopi, Zuni, Navajo, and Jemez people of Arizona and New Mexico. This region of the United States is home to a rich and diverse array of different cultures, each having their own traditions, languages, and forms of artistic expression. The people of the American Southwest are working to integrate their native practices with modern technology. Our examination of their pottery, a product of their deep cultural traditions, brought us to consider several deep mathematical questions.

Family Circle: Star Polygons

We began our exploration with this pot by Fannie Nampeyo, a famous Hopi potter. This pot has eight copies of the 'bear claw' motif laid out in a circle.

We chose the same point on each copy of the bear claw, then numbered them around the circle. Then we connected these points in two different ways. The blue lines connect each consecutive point, and form a regular octagon. The yellow lines connect every other point. Points 1, 3, 5, 7 form a square, but we have skipped over certain points. Connecting the points we had skipped gives us another square, and together they form a ‘star’.
We got a third star by connecting every third point. (In the diagram on the left we have ‘abstracted’ the diagram from the original pot). We noted that this last star polygon contains the other two within it.

If we connected every fourth vertex, we would merely get a diameter of the circle that describes the pot, skipping many vertices. If we repeat this procedure, we get an ‘asterisk’, which wasn’t really that interesting. Including the asterisk, we now have four stars.

**Question:** What would happen if we skipped more points? The answer is that we would get the same stars again, but traced in a different order around the circle.

**Question:** When we connected every second point around this circle, our polygon is not ‘connected’: we had to lift our pencil to draw the second square. But this is not true for the simple octagon, nor for the star polygon obtained by connecting every third point. When will our star polygon be connected? The answer is that it is connected when we connect every nth point, if n is relatively prime to 8.

Next we looked at a pot by the Zuni artist Anderson Peynetsa. It had seven figures of a deer spaced equally around the circle. Again, we took corresponding points on each deer figure and connected them to form a regular polygon, a 7-gon (heptagon).

And again, we connected every second point, every third point, etc. We found three different star polygons (if we include the original 7-gon among them). The diagram on the right shows the star drawn by connecting every third point, which includes the other two polygons within.

Note that since 7 is a prime number, every polygon we draw will be closed.

**Challenge:** Which star polygons include the others within? How shall we define the term ‘include’? How would we prove an assertion about one polygon including another? We noted only that although students quickly see what phenomena we are describing, they often have difficulty with a formal explanation of it.

**Challenge:** If we circumscribe each polygon in the diagram as shown, what is the ratio of the radii of the circumcircles?

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**Winding Numbers**

Start at any vertex of the simple star on the right (which is just a regular 9-gon), and ‘walk’ around it. Think of the center of the circle as a pole, and think of holding a ribbon attached to the pole.

As you walk around the polygon, the ribbon winds around the pole. When you have returned to the vertex you started from, the ribbon will have wound once around the pole.
Now start at a vertex of the star polygon shown on the left, and walk around until you come back to the vertex. This time, the ribbon will wind twice around the pole. We say that the ‘winding number’ or ‘density’ of this star polygon is 2. The winding number of the simple 9-gon is just 1.

**Challenges:**

1. What is the winding number for each of these star polygons?

2. How does the winding number relate to the size of each vertex angle?
3. What might it mean for the winding number of a star polygon to equal -2? Hint: we have only shown four polygons here.

We noted that the winding number is a *topological* property: if we moved the ‘pole’ around which we are winding away from the center of the circle, the number of ‘winds’ would stay the same, as long as we do not cross over a side of one of the polygons.

**An Intriguing Pot**

The bowl shown at left is by a Navajo artist. (The signature is difficult to read, and is given on the right.) The artist has painted a ten-point star on the bowl, but only shown the points of the star. She or he has also painted an 11-point star on the inner circle.

**Challenges:**

1. What kinds of stars are these?
2. How many points have been skipped around the corresponding circle in each case?

**Some More Advanced Ideas**

One of the joys of mathematics is that simple ideas quickly grow into profound and complicated ones. Here are some advanced ideas we can explore.

Take a circle. Fix a point A on the circle. We will start drawing polygons at A. Now choose a (random) point B for the second vertex of the polygon. Points A and B determine a polygon in the following way:

Find a third vertex C so that arc AB = arc BC.
Find a fourth vertex D so that arc BC = arc CD.

…and so on, until we get back to point A. The polygon may be a star polygon, or a simple regular polygon.
Challenges:

1. How can we figure out, from the positions of points A and ab, how many sides the polygon will have?
2. Or even if we will EVER come back to point A? Maybe we will go round and round, and never hit point A a second time. That is, the 'winding number' of our polygon will be infinite.

This last question is related to the notion of commensurability. Two quantities are commensurable if their ratio is a rational number (that is, if their ratio is the same as the ratio of two integers).

The polygon will close if the measure of arc AB is commensurable with that of a whole rotation (of the full circle).

If it is not, we have another question: Suppose point B is chosen so that the length of arc AB is not commensurable with a full rotation. Then we will get a ‘polygon’ with infinitely many vertices. How are these vertices distributed around the circle?

Will they eventually ‘fill in’ the circle? That is, will the set of vertices be ‘dense’ on the circle? If so, then a set of, say, a million vertices will look to the naked eye just like the circle—although if we think about it, the points P whose arc AP is commensurable with a full rotation will never be vertices.

OR will the set of vertices only be in select areas? For example, would a set of a million vertices look like this (all the red points, and no others):

This question turns out to unfold to become a fascinating and important theorem. It is closely related to the following intriguing question: Look at the first few powers of 2, listed on the left. It is not hard to see (and not even that hard to prove) that the rightmost digits form a cycle (after the very first): 2, 4, 8, 6, 2, 4, 8, 6... 

What about the leftmost digits? It turns out that they never form a cycle. If we went far enough, we could find a power of 7 starting with the digit 7. Or with the two digits 77. Or with the three digits 777....

We will not prove this here, but will give a sketch of how the proof might look. Suppose we want a power of 2 that begins with the digit 7. Then $7 \times 10^k < 2^n < 8 \times 10^k$. Taking logarithms (to base 10), this means that $k + \log 7 < n \log 2 < k + \log 8$.

Next we note that the leftmost digit of a number will not change if we multiply or divide it by 10. Division by 10 only affects the rightmost digits of a number. Divide a number by 10, and you subtract 1 from its logarithm. Since we can repeat this operation, we can subtract whatever integer we please from the inequality above. In short, this means that we need only look for a number $n$ such that $\log 7 < \{n \log 2\} < \log 8$, where $\{n \log 2\}$ means ‘the fractional part of $\log 2$’: the part of the number that remains when you subtract off the largest integer that’s smaller than the number.

And now we can connect this question to the question about polygons. Think of the circle as formed by wrapping the unit interval $0 \leq x < 1$ around its circumference, so that each point on the circle has a label (a coordinate) which is a real number between 0 and 1. Then choose point B so that its coordinate is log 2 (which is the same as its fractional part, because it is in the unit interval). Forming the sequence $2 \log 2$, $3 \log 2$, $4 \log 2$... just means finding points C, D, E... as we did around the circle. And we are asking whether one of the points in this sequence lands between log 7 and log 8.
It turns out that this must always happen: the fractional part of integer multiples of *any* irrational number are *dense* in the unit interval.

So: (1) wherever we pick point B in constructing our polygon, either the polygon will close exactly at point A, or will have vertices that ‘fill up’ the whole circle (or nearly so).

And: (2) whatever integer T we might pick (not just 2), we can find an integer $T^n$ ($n = 1, 2, 3, \ldots$) that starts with any preassigned sequence of digits.

For teachers: You may find useful the Powerpoint presentation at https://docs.google.com/presentation/d/1dKlaFK3tyW2ksa4zDnRqD1jgvxdX_mH1/edit#slide=id.p1 (English) or https://docs.google.com/presentation/d/1xN89VQHeN-TVnaxteQXHVbmXVGNcjit9/edit#slide=id.p1 (Ukrainian)

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