Introduction

This was the first live meeting of the Sunflower Bluebird Math Circle, and we are excited to report that it was attended by both students and teachers. Thank you so very much for everyone who came and participated in producing beautiful math – your enthusiasm and insight were electrifying! People not only enjoyed playing with triangles, but also made interesting and far-reaching observations and offered bold conjectures. That’s exactly what we hope for every Sunflower Bluebird meeting.

Kateryna Terletska started the meeting by brief explanation of the program and short introduction of the session leaders, Tatiana Shubin and Maria Droujkova. Then the fun began – and it went really well thanks to brilliant translation by Katia Antoshyna.

First, we looked at the picture of three traditional Pomo baskets.

These baskets were made of willow sticks or reeds. Amazingly, the crafters wove them so tightly that people used to cook soups and stews using the baskets as pots. People who created the baskets tried to make them not just useful but also beautiful. They adorned them with various designs. All three baskets in the pictures shared one particular design element. We asked the participants to identify it and got an immediate response – it’s a triangle. And we proceeded to deal with triangles.

Warm-Up Activities

1. Which of the two triangles - blue or yellow - has a larger area? Notice that bases BC and DE have equal length. You may also notice that the triangles share vertex A, and that they also share the height AH.

People were very quick in saying that the triangles ABC and ADE had the same area. And they further explained that the area of a triangle is equal to half the product of its base and the corresponding height. Thus the area of triangle ABC is $\frac{1}{2} \times BC \times AH$, while the area of triangle ADE is $\frac{1}{2} \times DE \times AH$, and these values are the same since BC=DE.

Of course, this is right. But there is a different and more visual way to see it – we can cut a triangle into pieces and then rearrange the pieces so that they form a rectangle whose base is the same as the triangle’s base and whose height is exactly half of the triangle’s height. Of course, the area of a rectangle is the product of its dimensions. Here is one way to do it. We start with triangle ABC and take the following steps:

a. Divide the height AH into two equal parts by point M, so that AM=MH. Draw a line parallel to the base BC through point M. The line intersects AB and AC at points P and O, respectively.
b. Cut the triangle along the line PO. Place triangle APO so that point A coincides with point C, and line segment AO coincides with CO. (In other words, rotate triangle AOP through an angle of 180 degrees around point O.) We have rearranged the parts of ABC into a parallelogram PBCQ. Obviously, it has the same area as triangle ABC. It also has the same base BC, and its height is exactly half of the height AH.

c. Draw a line segment PT so that PT is perpendicular to BC. Cut out the triangle PBT and translate it along the base so that its edge PB coincides with QC. We have constructed a rectangle PTSQ. It has the same area as triangle ABC. It also has the same base since TS=BC, and its height is exactly half of AH.

2. **Dissect a square into a number of triangles of the same area:**

A. 2 triangles
B. 3 triangles
C. 4 triangles
D. 6 triangles
E. 10 triangles

2.A. Dividing a square into 2 triangles of the same area was easy – everyone used a diagonal.

2.C. Likewise, everyone used both diagonals of a square in order to divide the square into 4 triangles of the same area.

2.D. This part was much more challenging – and interesting. Here people had a number of different solutions; we saw at least the following five. Notice that in both (i) and (ii) all the triangles are congruent. While triangles in each of cases (iii) – (v) have the same area (recall Warm-Up activity 1), they aren’t congruent.
What was most mathematically promising here was the observation – almost immediately pointed out by many people, for example, Katya, Anastasiya, Denis, Anna, Andrij – that using exactly the same approach one can cut a square into any even number $2n$ of triangles of the same area. For example, divide the square into $n$ equal rectangles, then split each of those rectangles by a diagonal (as in (i) or (ii)). Or divide two opposite (iii) or adjacent (iv) sides of the square into $n$ equal line segments each and connect those points to two (iii) or one (iv) vertex of the square. Or draw a diagonal, divide it into $n$ equal pieces and connect their end-points to two opposite vertices of the square (v). Can you find any other ways? Share your solution with the Sunflower Bluebird.

But what about part 2.B.? Can we split a square into 3 triangles of the same area? Many people - for example, Kiril, Katya, Mariya, Denis, Andrij, Volodimir, Vladislav - said that it wasn’t possible. Moreover, some said that it isn’t possible to split a square into any odd number of triangles of the same area. And those people were exactly right! Yes, it’s a fact. An amazing thing is that to prove this general statement is not easy – we’ll talk about it later. Before reading on, try to prove the impossibility of cutting a square into 3 triangles of the same area. And share your proof with the Sunflower Bluebird!

Share your ideas with other Sunflower Bluebird Math Circle participants at https://www.facebook.com/SunflowerBluebird

The Triangle Game

The Triangle Game is played on a equilateral triangle board. Vertices of the triangle are labeled counterclockwise by three colors which we denote by 1, 2, and 3 for convenience. Also, parallel to each side, three equally spaced lines are drawn across the triangle, thus creating 16 small triangles.

The game is for two players, who take turns to label the unlabeled vertices of the figure, in accordance with the following rules:

- a vertex on edge $\{1,2\}$ may be labeled either 1 or 2, but not 3
- a vertex on edge $\{2,3\}$ may be labeled either 2 or 3, but not 1
- a vertex on edge $\{3,1\}$ may be labeled either 3 or 1, but not 2
- a vertex inside the big triangle may be labeled 1 or 2 or 3.
When all the vertices have been labeled, the scores of the two players are calculated as follows:

- The score of Player 1 is the number of small triangles which are labeled \(\{1,2,3\}\) counterclockwise.
- The score of Player 2 is the number of small triangles which are labeled \(\{1,2,3\}\) clockwise.

The winner is the player with the higher score.

We started by demonstrating an example of the game on the screen.

Then two volunteers – Sergey and Kiril - played against each other. They took turns calling out their moves, while the leader moved the tokens for them, screen-sharing for everyone. Of course, each tried to beat their opponent. Eventually, Sergey (Player 1) won. If you'd like to run a similar activity, you could copy our demo jamboard here: [http://tinyurl.com/SBB1-jam](http://tinyurl.com/SBB1-jam)

Then it was time for all people to try their hand at the game. They used an interactive [https://scratch.mit.edu/projects/707742053/fullscreen/](https://scratch.mit.edu/projects/707742053/fullscreen/) Each person played for both Player 1 and Player 2. We asked them to keep their scores. While playing, people made keen observations and formulated some general ideas:

- Player 1 always wins (Andrey, Anastasiya, Sergey, Darya, Oksana, Maksym, and others).
- It's impossible to have a draw (Kiril, Svyatoslav, Mariya, and many others)
- Player 1 always has one more triangle than Player 2 (Oleksander, Mariya, Svyatoslav)
- The total number of \{1,2,3\} small triangles is odd (Anna)

Mariya pointed out that the player who counts those small triangles that have the same orientation as the original triangle always wins.

Then we recorded the scores in the following table:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Player 1 triangles</td>
<td># of Player 2 triangles</td>
<td>Winner: P1 or P2</td>
<td>Difference?</td>
<td>Sum?</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>4</td>
<td>P1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>3</td>
<td>P1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>P1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>P1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>4</td>
<td>P1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>2</td>
<td>P1</td>
<td>1</td>
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<tr>
<td>8</td>
<td>4</td>
<td>3</td>
<td>P1</td>
<td>1</td>
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<tr>
<td>9</td>
<td>3</td>
<td>2</td>
<td>P1</td>
<td>1</td>
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<tr>
<td>10</td>
<td>1</td>
<td>0</td>
<td>P1</td>
<td>1</td>
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<tr>
<td>11</td>
<td>3</td>
<td>2</td>
<td>P1</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>2</td>
<td>P1</td>
<td>1</td>
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<td>0</td>
<td>P1</td>
<td>1</td>
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<td>4</td>
<td>3</td>
<td>P1</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>3</td>
<td>2</td>
<td>P1</td>
<td>1</td>
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<tr>
<td>17</td>
<td></td>
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</tr>
</tbody>
</table>

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The table is striking, isn’t it? We see that in all the recorded cases Player 1 always wins, and his/her score is always exactly one more than the score of Player 2. It can’t be just a coincidence! And it isn’t – it’s the result of Sperner’s Lemma (in what follows we’ll refer to it as SL).


Divide a triangle \( T \) into lots of small triangles, so that small triangles only meet at a common edge or a common vertex (we call such division triangulation). Label each vertex of the whole triangle by 1, 2, or 3; then label vertices on the \( \{1,2\} \) side by either 1 or 2, on the \( \{2,3\} \) side by either 2 or 3, and on the \( \{1,3\} \) side by either 1 or 3. Label the points in the interior by any of 1, 2, or 3. We’ll call it a Sperner-labeled triangulation.

Any such labeling must contain a small \( \{1,2,3\} \) triangle. In fact, there must be an odd number of these triangles. Moreover, the number of \( \{1,2,3\} \) triangles with the same orientation as the original triangle is always one more than the number of \( \{1,2,3\} \) triangles of the opposite orientation (e.g., if the labels 1, 2, 3 of the original triangle are placed counterclockwise, then for any labeling there will be exactly one more small triangles with labels 1, 2, 3 placed counterclockwise than those small triangles where labels 1, 2, 3 are placed clockwise).

The above two paragraphs describe SL in two dimensions. In fact, a version of the lemma holds in all dimensions! In particular, the 1-dimensional case goes as follows:

Suppose that we have a finite number of points on a line segment. Let’s label the first point 1 and the last point 2. Then no matter how we label the rest of the points, there will be an odd number of small segments with endpoints 1 and 2.

Try to prove this 1-dimensional case on your own. Hint: You might use mathematical induction on the number of points. Write to the Sunflower Bluebird if you need help or want to share your proof.

There are many different proofs of the lemma. Below is a proof by Alexander Adam Azzam https://adamazzam.wordpress.com/2012/05/18/sperners-lemma/

Proof: Let \( T \) be a triangle with a Sperner-labeled triangulation. Think of the triangle as a house, triangulated into triangular rooms. Let’s call every line-segment in the subdivision labeled \( \{1,2\} \) or \( \{2,1\} \) a “door” (shown in red below).

Notice that a fully labeled (i.e., \( \{1,2,3\} \)) small triangle only has one door and any other small triangle with a door must have two doors. This is because any room with at least one door has either no repeated labels (it is fully labeled), or it has one repeated label that appears twice. Since the triangulation is given a Sperner labeling, the only doors into the house from the outside are on the side (boundary edge) whose corners are labeled 1 and 2. Thus, by Sperner’s Lemma for the 1-dimensional case, we know that there are an odd number of doors that lead from the outside to the inside.

Let’s start at any of the doors on the boundary and walk inside through the door. Either the room you’re in has another door for you to exit through, or there are no other doors. If there are no other doors, then you’re standing in a room with one door. So you’ve found a completely labeled room and we’ve proven the existence of a fully labeled triangle. Otherwise, continue walking through doors into the one adjacent room, repeating this process (convince yourself why we can never double back on a room). Since the number of rooms is finite, the procedure must terminate and so at the end you’ve either found yourself outside or stopping somewhere inside. If you stop inside, then by the previous argument you’ve found your completely labeled room.
If you stop outside, then you’ve just paired up precisely two doors on the boundary of T that do not lead to a completely labeled room. However, since the number of boundary doors is odd, there must be at least one door that leads, and stays, inside. This proves the existence of at least one completely labeled room.

In fact, this shows that there are an odd number of boundary doors which are the beginning of paths that terminate at a completely labeled room inside. Moreover, any completely labeled rooms not reachable by paths from the boundary must come in pairs – we can start at any such room and then repeat the same process of walking about, which must terminate at some other completely labeled room.

Let’s summarize our findings. There are an odd number of fully labeled rooms reachable from the outside doors. If there are fully labeled rooms unreachable from the boundary there are an even number of them. Thus, the total number of completely labeled rooms is odd.

It’s a nice proof, but it doesn’t seem to settle one statement that we made: the number of fully labeled triangles with the same orientation as the original triangle is one more than the number of fully labeled triangles with the opposite orientation. So here is another proof, by Mel Currie (from his book “Mathematics: Rhyme and Reason”). Before we come to his proof, we’d like to provide a quote from Mel’s book where he says: “I do not remember when I first encountered Sperner’s Lemma. I do know that it was when I was well beyond the nursery-rhyme phase of my mathematical development. I wish I had bumped into it much earlier. It’s a real delight.”

**Mel Currie’s Proof:** In this proof, instead of using labels 1, 2, and 3, we will use labels A, B, and C. Also, we will call triangles with all three labels ‘complete triangles.’ Let’s suppose that the original triangle has the counterclockwise orientation: the motion from the vertex A along the edge AB towards vertex B, then along the edge BC towards vertex C is counterclockwise. We will say that this orientation is +1. If we imagine walking the perimeter counterclockwise, the interior of the triangle will always be on our left.

If the motion around the perimeter, vertex A to B to C, is clockwise, the orientation is said to be -1. It is useful to denote the orientation of the edges AB, BC, and CA to be +1 if the triangle ABC has a counterclockwise orientation and -1 otherwise. That is to say that the triangle’s edges inherit the orientation of the triangle in a straightforward way.

Now consider the triangle below. We have subdivided edge AB by adding a point and labeling the point A.

Our original AB edge now consists of two segments, AA and AB. We define the orientation of the segment AB to be the same as the orientation of the edge if the direction A to B on the segment is the same as the direction from A to B of the vertices A
and B. We see that the direction of the segment AB is the same as the direction of the directed line segment from vertex A to vertex B. So both edge and segment have orientation +1. Similarly, if we subdivide our edge AB by placing a B label between A and B, we are again left with only one segment AB with the +1 orientation.

In general, we define the **Index** of the edge AB, which we have subdivided, to be the sum of the orientations of the segments that have both labels, A and B. For example, the Index of the subdivided edge below is +1.

We see this by moving down the edge from right to left: we have an AB segment with orientation +1, BA segment with orientation -1, and AB segment with orientation +1. The final segment, BB, does not contribute to the Index. The Index, the sum of the orientations, is +1.

It is now straightforward to see that the Index of any subdivision of AB will always be the same as the orientation of the edge AB, which is the same as the orientation of the original triangle ABC. The proof is by induction on the number of labels that have been used to create the subdivided edge. We have effectively shown above that if we just insert one point of either letter (A or B) our claim is true. Now assume that we are subdividing edge AB and our claim holds for the first n labels that we have inserted. If we insert one more label A, it must be between a pair of labels, BB, AB, BA, or AA. Inserting the label A on the segment BB gives rise to BA and AB. They have opposite orientations, so their signed orientations cancel and the Index with the additional point remains unaltered. Inserting the label A on segments with endpoints AB or BA simply replaces the old segment with a new AB or BA, respectively, along with an AA segment. No change. Of course there is no change in the value of the Index if we insert an A on AA. The argument that unfolds when we add the label B is completely analogous.

We can “sum things up” as follows. Anytime we apply a labeling to the edges of a triangle ABC à la Sperner, the Index of the edge AB will coincide with the signed orientation of the triangle, +1 or -1. It should be clear that there is nothing special about the edge AB, since edges AC and BC have the same orientation.

As a reminder, our goal is to show that when we triangulate the triangle ABC and do a Sperner labeling, we produce at least one triangle that has a vertex with each of the three labels. Further, we want to show that we will have one more complete triangle with the orientation of the outer triangle than with the opposite orientation. To this end, we introduce another definition. Let the **Content** be the sum of the (signed) orientations of the complete triangles in the triangulation.

If we can show that the Content equals the Index, we’re done. Before we take this on, we have just one more definition. Let $S$ (for *substance*) be the number of segments labeled AB in the triangulation counted by adding orientation values (+1 or -1), including those on the original triangle. If the segment is in the interior, it is counted twice, once for each of the triangles to which it belongs. We will finish the proof by showing that both the Content and the Index equal $S$. 
First, we show that the Content equals $S$.
What does the complete triangle contribute to the Content? Since the orientation is counterclockwise, the answer is $+1$. But this is exactly what it contributes to $S$, since it only has one counterclockwise AB edge.

Now we are on a roll and quickly see that the complete triangle with clockwise orientation also contributes the same quantity $(-1)$ to both $S$ and the Content. Triangles that do not have both an A and a B in their labeling contribute zero to $S$ and zero to the Content, since they are not complete. Finally, triangles that have two Bs and an A or two As and a B have AB edges with different orientations and contribute zero to $S$ and, not being complete, contribute zero to the Content. So, the Content equals $S$!

Show that the Index equals $S$.
Well, any AB edge in the interior contributes zero to $S$, since it will appear twice with the opposite orientation. It follows that interior edges contribute a net of zero to $S$. That leaves only the exterior AB segments to determine the ultimate value of $S$, whose signed sum also determines the Index. The Index equals $S$! Suddenly we’re done. **Index equals Content!**

We have shown that for any triangle the Content of a Sperner-labeled triangulation can only be -1 or 1, so there must be at least one complete triangle, otherwise the Content would be zero. Further, the triangulation must have one more complete triangle with the same orientation as the original triangle than with the opposite orientation, since the Index equals the signed orientation of the triangle.

This concludes the proof of Sperner’s Lemma. We are done!

And now we see that The Triangle Game wasn’t fair! No matter what players do, it’s inevitable that Player 1 will win. Not even a draw is ever possible (confirming the observation made by many participants). And there is no winning strategy (except, as Darya, Vladislav, Anastasiya, Maksym, Mariya, Kiril, and others noted, for making sure that you are Player 1!). So much for the Triangle Game.

It's really amazing how powerful Sperner's Lemma is. There are many applications of the lemma. In particular, do you still remember our Warm-Up activity 2? And the conjecture that it is impossible to cut a square into an odd number of triangles of the same area? Well, mathematicians had suspected this for a long time but couldn’t prove it until the first proof was found in the 1970’s - and it was based on Sperner’s Lemma.

Another astonishing fact is that Sperner’s Lemma is equivalent to the Brouwer Fixed Point theorem, a theorem of algebraic topology that was stated and proved in 1912 by the Dutch mathematician L.E.J. Brouwer. When restricted to the one-dimensional case, Brouwer’s theorem can be shown to be equivalent to the Intermediate Value theorem, which is a familiar result in calculus. As a practical application we included it in the newsletter - as a **Fun Fact of the Fortnight**: 

Pour yourself a cup of coffee. Stir it gently for a few seconds, then wait for it to settle. Inevitably, there will be at least one molecule somewhere in the drink that ends up in exactly the same location it was before you started stirring! It might have moved in between, but it comes back to its original position in the end.

Submit your math-related questions at [https://www.facebook.com/SunflowerBluebird](https://www.facebook.com/SunflowerBluebird)