# BLUEBIRD MATH CIRCLE
# Alliance of Indigenous Math Circles

## Issue 30 Recap: I Spy

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https://aimathcircles.org/Bluebird

**NEWSFLASH** Join LIVE Bluebird Math Circle with friends and family.

Monday, July 11, 5-6 PM MDT online.

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**https://aimathcircles.org/Bluebird**

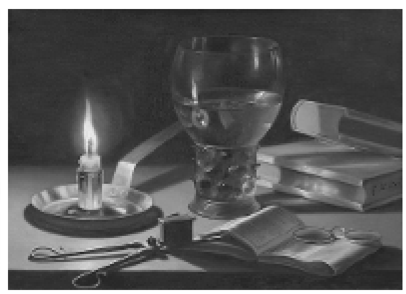
## Introduction

As usual, we started by thinking about questions for the Bluebird while listening to a bluebird singing his beautiful song. And people asked some good questions which appear at the end of this recap. Then Donna Fernandez led us to the next part of the session.

## Inspiration: Native American and Old European Art

Donna invited participants to look at the three pictures below and share their thoughts about possible 'hidden messages' there. She said, "Traditional art represents beliefs and is culturally significant. Can we discern beliefs or ideas in each of the three pictures below? What do *you* see?"





Some answers were:

> *I see regular figures, which I think of as asserting regularity and harmony in life. And the baskets are certainly not European, as they have 10-fold rotational symmetry. Almost all European art has rotational symmetry in the form of* $2^n$ *: 2, 4, 8, 16, and so forth.* – Mark Saul

And about the last picture people noticed:

> *Maybe the picture shows different forms of 'enlightenment'* – Mark Saul

> *I just noticed that the candle and the water are half-done. And one book, too!* – Maria Droujkova

Donna pointed out some ideas mentioned in the newsletter which we're copying below, and asked people to share any additional thoughts with the Bluebird at https://aimathcircles.org/Bluebird

1. Hopi plaque, artist unknown. Baskets are made from plants establishing a connection to the land. "Plaques play an intricate role in Hopi society and are given by women as thank you gifts, as well as by Katsinam. These baskets are also used during ceremonies and play an especially important role in the initiation ceremonies of Hopi girls."
https://blog.kachinahouse.com/a-history-of-hopi-basketry/

2. Navajo Wedding Basket, artist unknown. All elements of the design have important meaning. For example, the outer band signifies People, Animals and Plants; next bands going in symbolize the Path of the Sun, then Path of the Moon, and Path of

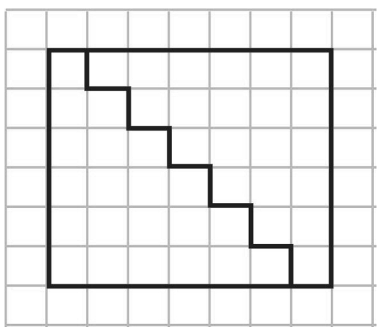
the Constellations. Other elements include Earth, Different Types of Mountains, Sunray and Rainbow, Clouds and Different Types of Rain, Place of Emergence, Holy People, East and Dawn. Detailed explanation: @nizhinibahnavajo on Instagram https://www.instagram.com/p/7yhPV7O99o/

3. Pieter Claesz, Still Life with Lighted Candle, 1627, Mauritshuis, The Hague, Netherlands. One of the possible messages sent to us by the artist: The time of life is flitting, hurry up to acquire knowledge. Traditionally, a candle had lots of meanings. One meaning was the passing of time. Books are easy: they meant learning or transmitting knowledge.

Donna went on to say, “So that became our inspiration today, looking at these baskets and the old painting and seeing that they are beautiful, but they have meaning within them, as well. We hope that those three pieces will give us a good message about at was ‘hidden meaning’ is held in the diagrams we’ll be looking at next.”

## Warm-Up: Seeing in both senses (What Do I Spy?)

1. ***What do we see?***
2. ***Can we count the little (grid) squares inside the rectangular grid? What is the easiest way to calculate it?***
3. ***What is the number of grid squares under the ‘staircase’? (Hint: write an expression but don’t evaluate it.)***
4. ***What is the number of grid squares above the staircase? (Hint: write an expression but don’t evaluate it.)***
5. ***What do we see?***



1. The first thing that springs out of the picture is the staircase. And the figures below and above the staircase are congruent. We also see a rectangle which contains the staircase.
2. The easiest and fastest way to count the grid squares is by means of multiplication, as Mythili pointed out. The number is
3. $6 \cdot 7 = 42$.
4. It is $1 + 2 + 3 + 4 + 5 + 6$ .
5. As we saw above, the figures below and above the staircase are congruent, and therefore they must contain the same number of grid squares. Hence the answer here is again $1 + 2 + 3 + 4 + 5 + 6$.
6. We see that we can find the total number of grid squares in the rectangle in two different ways: as the product rectangle’s dimensions, and as the sum of the squares below and above the staircase. Hence using the expressions that we found above, we can see that $2(1 + 2 + 3 + 4 + 5 + 6) = 6 \cdot 7$. If we divide both sides of the equation by 2, we get a pretty interesting result:

$$1 + 2 + 3 + 4 + 5 + 6 = \frac{6\cdot 7}{2}$$

Nigel Wilson recognized this as a particular case of a formula for the sum of the first n positive integers:

$$1 + 2 + 3 +\ldots + n = \frac{n(n+1)}{2}$$

Nigel reminded us about of a famous story of Carl Gauss who found this formula when he was a very young child. Here's a version.

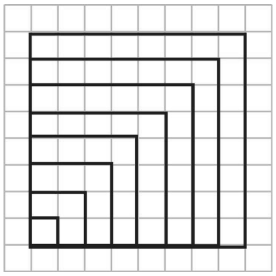
*Every great mathematician was, at one point, in somebody’s elementary school math class.*
*There is a legend that when the someday-to-be great mathematician, Carl Friedrich Gauss, was a child he would always finish his assigned work before his classmates and once he had he would cut up and distract them.*

*Frustrated, his teacher thought up a problem to keep him busy for a long time. He assigned little Gauss to add up all the counting numbers from 1 to 1000! To do this as a 999 step addition problem would surely take all day! But young Gauss found a brilliant short cut that only took minutes.*

We were ready to break into smaller groups and investigate other pictures—every time keeping in mind an important idea that **if we calculate the same thing in two different ways the result must be the same.**

## What Do I Spy?

1. ***Let's count grid squares.***



**A.**

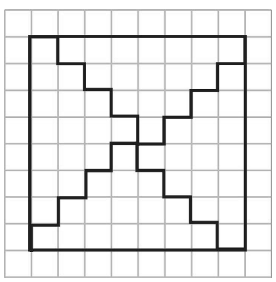
We can count the grid squares by 'layers' shown in the picture - start with 1 square at the bottom left, then adding 3 squares in the next L-shaped layer, then adding 5 squares in the next layer, and so on. Thus the total number of grid squares in the 8-by-8 square is $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$.

On the other hand, we can find this number simply as the product $8 \cdot 8$, or written in a different form, $8^2$ .
***We spy*** a wonderful result:

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 8^2$$

It's even more striking if we say it as follows: ***the sum of the first 8 odd numbers is 8 squared.*** Nice, isn't it?
It then stands to reason that **the sum of the first *n* odd numbers is** $n^2$. The $n^{th}$ odd number is $(2n - 1)$ (check it!).
Thus we have the following general formula:

$$1 + 3 + 5 + ... + (2n - 1) = n^2$$



**B.**

The total number of grid squares inside the large 8-by-8 square is the sum of the squares in four 'pyramids'. Each pyramid has $1 + 3 + 5 + 7$ squares, hence all four of them contain the total of $4(1 + 3 + 5 + 7)$ squares.

On the other hand, this total number is simply $8^2$ . Hence we spy the equality:

$$4(1 + 3 + 5 + 7) = 8^2 .$$

In words: *4 times the sum of the first four odd numbers is eight squared*.
What happens if we add one more layer at the bottom of each pyramid? We'll get an equation

$$4(1 + 3 + 5 + 7 + 9) = 10^2 .$$ (Check it!)

In words: *4 times the sum of the first five odd numbers is ten squared.*

In general, it will read: *4 times the sum of the first n odd numbers is 2n squared.* Written as a formula:

$$4(1 + 3 + 5 + ... + (2n - 1)) = (2n)^2 .$$

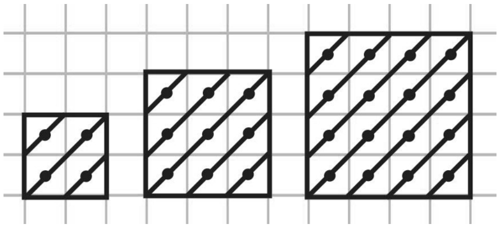
Let's simplify this last equation:

$4(1 + 3 + 5 + ... + (2n - 1) = 4n^2$ , and dividing both sides by 4 we get

$$1 + 3 + 5 + ... + (2n - 1) = n^2 ,$$

the same equation as in the part A above, only obtained in a different way!

***2. Let's count the dots.***



Start with the left picture. If we count the dots going diagonally, we get $1 + 2 + 1$. We can also count them as $2^2$.

In the middle picture, counting the dots diagonally we have $1 + 2 + 3 + 2 + 1$. We can also count them as $3^2$.

In the right pictura, counting the dots diagonally we have $1 + 2 + 3 + 4 + 3 + 2 + 1$. We can also count them as $4^2$. Thus we spy:

$$1 + 2 + 1 = 2^2$$

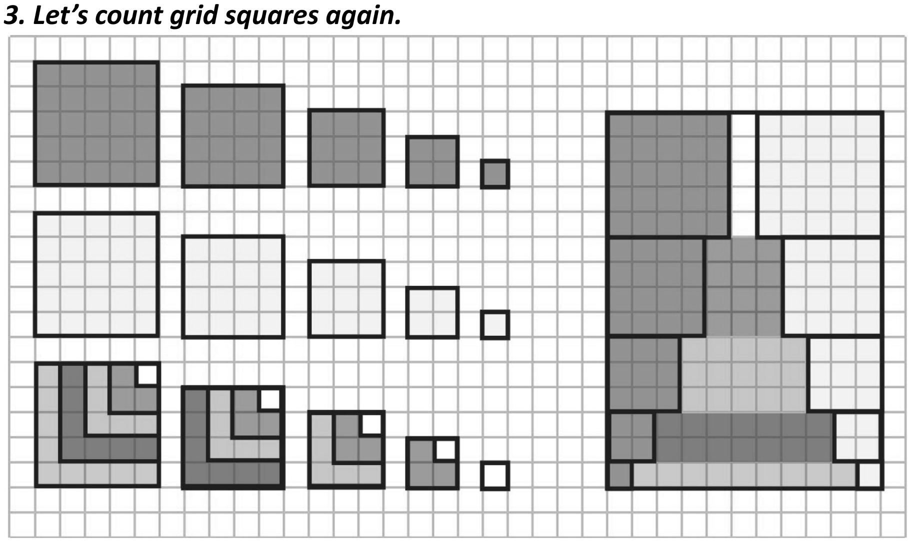
$$1 + 2 + 3 + 2 + 1 = 3^2$$

$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$$

In general:

$$1 + 2 + 3 +... + (n - 1) + n + (n - 1) +... + 3 + 2 + 1 = n^2$$

***3. Let's count grid squares again.***



The picture has two sides: on the left, we have a row of blue squares, a row of yellow squares, and a row of squares with bands of different colors. On the right, we have a rectangle. To begin with, we spy that we can use all the squares on the left to construct the rectangle on the right: we stack up blue and yellow squares to form the borders, then we proceed to disassemble and rearrange banded squares. First, we take 9 dark yellow squares and place them at the bottom (between blue and yellow 1-by-1 squares). Then we take two purple bands of 7 squares each from the first and second banded squares and place them right on top of the dark yellow squares. Then we take three green bands, each of 5 squares, and put them on top of the purple squares. We continue taking four orange bands of three squares each and placing them on top of green squares. We are left with five white squares, and we finish our construction by stacking them up just filling in the gap between blue and yellow 5-by-5 squares. Hence we see that the total number of the grid squares in the three rows of squares on the left is the same as the number of grid squares in the rectangle on the right.

Now, let's count the number of grid squares in the row of blue squares. It's $1^2 + 2^2 + 3^2 + 4^2 + 5^2$.

The number of grid squares in the row of yellow squares is the same. Likewise, it's the same in the row of banded squares.

Therefore, the total number of the grid squares in the rectangle on the right is $3(1^2 + 2^2 + 3^2 + 4^2 + 5^2)$.

We can also find the number of grid squares in the rectangle as the product of its dimensions. We see that its height is $(1 + 2 + 3 + 4 + 5) = \frac{5\cdot6}{2}$ (recall the formula we found in the warm-up activity!). We can only see that its width is $(2 \cdot 5 + 1)$ - look at the top edge.

So we spy:

$$3(1^2 + 2^2 + 3^2 + 4^2 + 5^2) = \frac{5\cdot6}{2} \cdot (2 \cdot 5 + 1) .$$

Dividing both sides by 3 and simplifying the equation we obtain the following equation:

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \frac{5\cdot6\cdot(2\cdot5+1)}{6}$$

Can we generalize this formula? You bet! Just imagine that we have three rows of blue squares, yellow squares, and banded squares, with squares of sizes from 1-by-1 up through n-by-n in each row. Then construct a rectangle using the same procedure we did. The rectangle will have the height of $1 + 2 + 3 +\ldots+ n = \frac{n(n+1)}{2}$ and the width of 2n+1. Hence

$$3(1^2 + 2^2 + 3^2 +\ldots+ n^2) = \frac{n(n+1)}{2} \cdot (2n + 1)$$

Now, dividing both sides by 3 and simplifying we get a pretty impressive result:

$$1^2 + 2^2 + 3^2 +\ldots+ n^2 = \frac{n(n+1)(2n+1)}{6}$$

Share your ideas with other Bluebird Math Circle participants at **https://aimathcircles.org/Bluebird**

## New Questions for Bluebird

*How fast is the fastest bird?* — from Mythili P

*What is the hottest temperature a human can withstand?* — from Donna Fernandez

*When we compose two rotations around the same center, it's easy to see that we get another rotation around that center. But what if we compose two rotations around different centers?* — from Mark Saul


**BLUEBIRD SAYS**—Fascinating questions! I will fly around and seek answers. Watch this space in the next newsletter issues!

Submit your math-related questions at **https://aimathcircles.org/Bluebird**