# BLUEBIRD MATH CIRCLE
# Alliance of Indigenous Math Circles

## Issue 28 Recap: Fair and Square

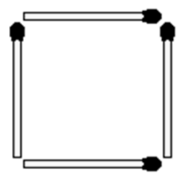
Share your problems, solutions, models, stories, and art:
https://aimathcircles.org/Bluebird

**NEWSFLASH** Join LIVE Bluebird Math Circle with friends and family.

Monday, June 6, 5-6 PM MDT online.

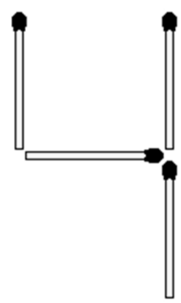
Sign up at
**https://aimathcircles.org/Bluebird**

## Math Puzzle



*The task we had was: Move one matchstick to create a different square*

**Solution:**
If we take the top matchstick and place it vertically right below the right vertical stick, we get the following figure:

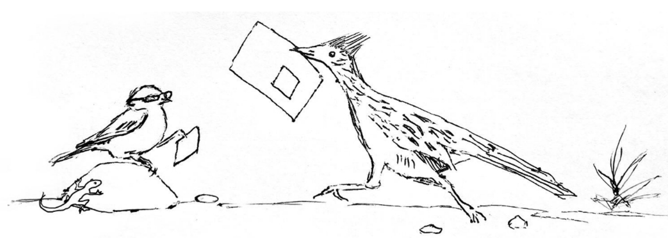


And this is number 4, which is a square: $4 = 2 \times 2 = 2^2$.
(BTW, square numbers will play an interesting role in the further activities.)

This puzzle is an example of so-called lateral thinking **https://en.wikipedia.org/wiki/Lateral_thinking** Math people cultivate these puzzles because they help to nurture insight, divergent thinking, and seeing unexpected connections. And they are good fun!

## Warm-Up Activities

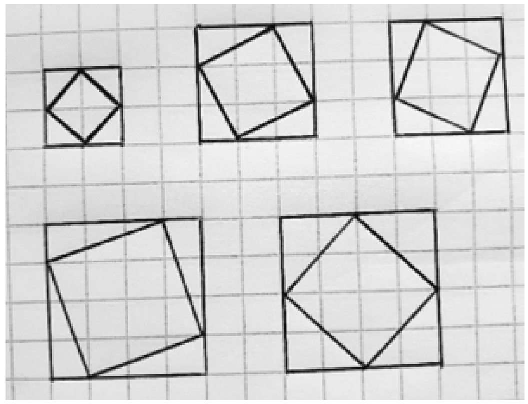
**1**. *One nice summer afternoon Bluebird came to see his friend Roadrunner and was greeted with a question. For his handicraft project, Roadrunner needed a perfect square made of cardboard. He had taken a piece of cardboard and cut out the desired shape. Now he had the cardboard with a hole in it and the cut-out piece, and he wanted to verify that the piece was indeed a square, but there were no tools whatsoever – neither a ruler, nor a compass, nor anything else. Bluebird solved the problem! Can you solve it, too?*



**Solution:**
First of all we recalled what a square is: a flat figure with 4 sides and 4 angles such that all sides are equal to one another, and all angles are equal, too. Hence we needed to check whether the cut-out piece has these properties. We realized that if

the cardboard was too thick we couldn't fold it in order to see if the sides and angles were equal in pairs. After some thinking, an idea jumped up: let's take the cut-out piece and try to fit it back in the hole. If it doesn't fit perfectly well we know that it isn't a square. But if it does fit, it might not be sufficient (can you see why?). So turn it around by 90 degrees and put it back in the hole. If it fits perfectly this time then it is indeed a square (why?).

**2.** *The area of each grid square is 1. Find the area of the inner shape in each of the pictures on the right. What are these figures?*



**Solution:**

Let's start with the top row and go from left to right:

1. The inside figure consists of four halves of grid square, and since the area of each grid square is 1, so each half has the area of ½. Thus four halves have the area of $4 \times 1/2 = 2$.

   There is another way which becomes very helpful when we try to find the areas of the other four figures. We notice that the regular square which the inner figure is inscribed in consists of 4 grid squares, and hence has the area of 4. To obtain the inner figure, we must cut out 4 triangles. Look at the two triangles on the left side: each one is a half of a grid square, so together they form a grid square with the area of 1. Likewise, the two triangles on the right also form a square of area 1. Hence the area of the inner figure can be calculated as $4 - 1 - 1 = 2$.

2. The regular square in which the inner figure is inscribed consists of $3 \times 3 = 9$ grid squares, hence its area is 9. To obtain the inner figure we need to cut out 4 triangles. Each of these 4 triangles is a half of a 2-by-1 rectangle, so two of them form such a rectangle, and the other two form another such rectangle. A 2-by-1 rectangle consists of two grid squares, so it has the area of 2. Hence the area of the inner figure is $9 - 2 - 2 = 5$.
3. The area of the inner figure is 5. It can be calculated in exactly the same way, or we can simply notice that this figure is exactly as the previous one, just tilted another way.
4. Similarly to what we did before, the area is $16 - 3 - 3 = 10$.
5. Likewise, the area is $16 - 4 - 4 = 8$.

The last question was: what are these inner figures? And the answer - not very surprisingly - is that each of them is a square. Indeed, in every case each side is a diagonal of a same size rectangle (a 1-by-1 rectangle for the first figure, a 2-by-1 rectangle for the second and the third figures, a 3-by-1 rectangle for the fourth one, and a 2-by-2 rectangle for the last one).

And what about the angles? For a square, all four angles must be the same. Can you see why this is true for each of the inner figures? Think about it and if you are not sure ask Bluebird!

Share your ideas with other Bluebird Math Circle participants at **https://aimathcircles.org/Bluebird**

## Family Circle: Let's Count

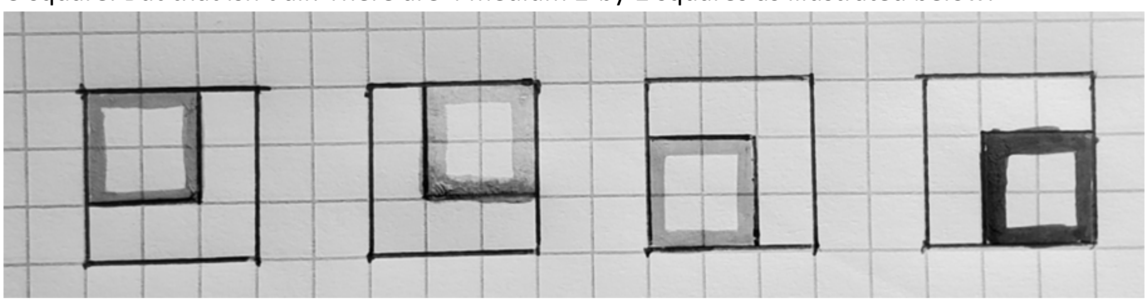
In this part we wanted to count regular squares (the squares with the sides on grid lines) and tilted squares (the squares whose vertices were at grid points, i.e., the points of intersection of grid lines, but whose sides do not lie on grid lines) in grids of various sizes.

We started with a 1-by-1 grid. Of course, everyone immediately saw that there is 1 regular and 0 tilted squares in there. It took us a little longer to count all regular squares in a 2-by-2 grid, but eventually we realized that there were 5 of them: four 1-by-1 squares, and one 2-by-2 square. And there was exactly one tilted square (like the one in the first picture in the warm-up activity about the areas above).

So we filled in the following table:

| The grid size | # of regular squares | # of tilted squares | Total number |
|---|---|---|---|
| 1-by-1 | 1 | 0 | 1 |
| 2-by-2 | 5 | 1 | 6 |

Next we looked at a 3-by-3 grid. This was trickier! We thought of all regular squares first. There are 9 small 1-by-1 squares, and 1 big 3-by-3 square. But that isn't all! There are 4 medium 2-by-2 squares as illustrated below:



So the total number of regular squares is 14, that is, $9 + 1 + 4$. Actually the three numbers in the sum are special - each one of them is a square (remember the math puzzle with matchsticks?). In fact, let's write this sum in a nice increasing order:

$1^2 + 2^2 + 3^2 = 14$ . This looks intriguing!

We were now ready to count all tilted squares. In fact, no 1-by-1 regular square contains a tilted one inside. Each of the four 2-by-2 squares (red, green, orange, and blue) contains exactly 1 tilted square. And the big 3-by-3 square contains exactly two tilted squares - see figures 2 and 3 in the warm-up activity about the areas. Thus, there are 4 + 2 = 6 tilted squares altogether. Hence we could enlarge and improve our table:

| The grid size | # of regular squares | # of tilted squares | Total number |
|---|---|---|---|
| 1-by-1 | $1 = 1^2$ | 0 | **1** |
| 2-by-2 | $5 = 1^2 + 2^2$ | **1** | **6** |
| 3-by-3 | $14 = 1^2 + 2^2 + 3^2$ | **6** | **20** |

We noticed two interesting patterns:

- The number of regular squares is the sum of consecutive squares starting with 1 and ending with the square of the grid size.
- The number of tilted squares is the same as the total number in the row above (look at the two blue 1's and two red 6's).

If these patterns are to hold (and yes, they are. The challenge is to justify this statement - try to do so!), then we can continue adding rows to our table forever. Let's add a couple of more rows:

| | | | |
|---|---|---|---|
| 4-by-4 | 30 = 14 + 16 | **20** | 50 |
| 5-by-5 | 55 = 30 + 25 | 50 | 105 |

The rest of the recap is for those who are into algebra: we will find the number of squares in an n-by-n grid for any number n. Let R(n) denote the number of all regular squares inside an m-by-m grid. Let T(n) denote the number of all tilted squares inside an n-by-n grid.

Let's place an n-by-n grid in the coordinate plane so that its vertices are (0, 0), (0, n), (n, n), and (n, 0). Every regular square inside this grid is of the size k-by-k, where 1 ≤ k ≤ n. For a given k, the top left vertex of a k-by-k square should be at a position (i, j) where 0 ≤ i ≤ n-k, and k ≤ j ≤ n, so there are exactly $(n - k + 1)^2$ regular squares of this size. Therefore

$$R(n) = (n - 1 + 1)^2 + (n - 2 + 1)^2 + (n - 3 + 1)^2 +...+ (n - n + 1)^2 = 1^2 + 2^2 + 3^2 +...+ n^2 = \frac{n(n+1)(2n+1)}{6}$$

Now we will find T(n). First, we observe that every tilted square is inscribed in a regular square. Moreover, each 2-by-2 regular square contains exactly 1 inscribed tilted square. Each 3-by-3 regular square contains 2 inscribed tilted squares, as we have observed (see, for example, figures 2 and 3 in the warm-up activity). In general, each k-by-k regular square contains exactly k-1 inscribed tilted squares. Therefore,

$$T(n) = (n - 2 + 1)^2(2 - 1) + (n - 3 + 1)^2(3 - 1) +...+ (n - n + 1)^2(n - 1) = (n - 1)^2 \cdot 1 + (n - 2)^2 \cdot 2$$
$$+ (n - 3)^2 \cdot 3 +...+ (n - (n - 1))^2 \cdot (n - 1) = (n^2 - 2n + 1^2) \cdot 1 + (n^2 - 4n + 2^2) \cdot 2 + (n^2 - 6n + 3^2) \cdot 3$$
$$+...+ (n^2 - 2(n - 1) + (n - 1)^2) \cdot (n - 1) = n^2(1 + 2 + 3 +...+ (n - 1)) - 2n(1^2 + 2^2 +...+ (n - 1)^2)$$
$$+ (1^3 + 2^3 + 3^3 +...+ (n - 1)^3) = \frac{n^2(n-1)n}{2} - \frac{2n(n-1)n(2n-1)}{6} + \frac{(n-1)^2n^2}{4} = \frac{(n-1)n^2(n+1)}{12}$$

Hence

$$R(n) + T(n) = \frac{n(n+1)(2n+1)}{6} + \frac{(n-1)n^2(n+1)}{12} = \frac{n(n+1)^2(n+2)}{12} = T(n + 1)$$

Therefore it's indeed true that the total number of squares in an n-by-n grid is exactly the same as the number of tilted squares in an (n+1)-by-(n+1) grid, precisely as we have observed!

Submit your math-related questions at **https://aimathcircles.org/Bluebird**