2.1 Grid Power

Contributed by Tatiana Shubin

Short Description: In this session, a simple sheet of grid paper is fertile
ground for generating questions that lead to combinatorial reasoning, finite
arithmetic series, algebraic identities, and the Pythagorean Theorem.

Materials: Participants will need lots of grid paper and pencils. You may
choose to use the included handout or you may simply write the questions
where participants can see them.

Mathematics Beneath & Beyond: This material is a treasure trove of
mathematical ideas. Among them:

We see combinatorics, or at least combinatorial reasoning (e.g., Solution 2 of
Problem 1). Along the way, we find sums of a finite arithmetic series, and
of the squares and cubes of integers. We discover some useful tools, such as
proving algebraic identities by means of representing quantities involved as ge-
ometric objects. In passing, we find out a fact that the slopes of perpendicular
lines are negative reciprocals of one another. And there is this astonishing but
undeniable phenomenon that an inquiry which started by simple counting led
us to geometry — we’ve proven the Pythagorean Theorem, felt the importance
and meaning of area as a measure (solving Problem 1), and had a chance to use
properties of parallel lines and similar triangles (using our approach to Problem
3).

Finally, there is number theory — while solving Problem 1, we run across two
peculiar sets of integers — one consists of integers which can be represented as
sums of two perfect squares, and another contains only integers lacking such
representation. Studying these sets can become a worthwhile endeavor which
would bring students quite deep into the fascinating field of elementary number
theory.
Grid Power Student Handout

Warm-up Problems:

(A) Is it possible to place dots into cells of an 8 × 8 square (no more than one dot per cell) so that the number of dots in every column is the same, while no two rows have the same number of dots?

(B) Is it possible to cover all squares of a chessboard with 32 dominoes (each domino covering exactly two squares) in such a way that no two dominoes cover a 2 × 2 square?
Problems:

1. (a) How many squares are there in a $7 \times 7$ square?

   (b) How many squares are there in an $n \times n$ square?

2. (a) How many rectangles whose sides lie on grid lines are there in a $7 \times 7$ square?

   (b) How many rectangles whose sides lie on grid lines are there in an $n \times n$ square?

3. Connect the vertices of a square in a cyclic way to the midpoints of the opposite sides. These four lines form an innermost quadrilateral inside the original square. What is it? What part of the area of the original square does it have? What happens if you replace the midpoint with a point that is a fraction $r$ along each side?
Grid Power Teacher Guide

Warm-up Problem Solutions:

(A) Since there are 8 rows and each row can contain no less than 0 and no more than 8 dots, the number of dots in each of 8 rows must be chosen from among the following numbers: 0, 1, 2, 3, 4, 5, 6, 7, and 8. So we can omit exactly one of them. Which one? The sum of all of them is

0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36

Since each column is to have the same number of dots, the sum of our chosen eight numbers (i.e., 36 minus the omitted number) must be divisible by 8, and only 4 does the job:

36 – 4 = 32 = 8 · 4

The diagram below shows a possible way of placing 0, 1, 2, 3, 5, 6, 7, and 8 dots so that each column has exactly 4 dots:
(B) We will try to avoid placing dominoes in a position covering a $2 \times 2$ square. First, let us label all squares of the chessboard with numbers from 1 to 64 as shown below.

Since all squares of the chessboard must be covered, a domino has to cover the bottom left corner, square 57. This can be done placing a domino vertically so that it covers squares 49 and 57, or horizontally, covering squares 57 and 58. Clearly, both situations can be analyzed in a similar manner; we will look at the former case:
We must also cover square 58, and the only way to do it and avoid covering a $2 \times 2$ square would be by placing a domino on squares 58 and 59. Now we need to cover square 50 avoiding covering a $2 \times 2$ square, which can only be done by placing a domino on squares 50 and 42.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
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<td>63</td>
<td>64</td>
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</tbody>
</table>

Continuing this process, we have no choice but to place the dominoes as shown below:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
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<td>60</td>
<td>61</td>
<td>62</td>
<td>63</td>
<td>64</td>
</tr>
</tbody>
</table>

Now we have to put a domino on squares 8 and 16, thus creating a $2 \times 2$
square. Therefore, it is impossible to cover the chessboard and avoid covering a $2 \times 2$ square with two dominoes.

Problem Solutions:

1. (a) There are 140 regular squares and 196 tilted squares for a total of 336 grid squares. See the Presentation Suggestions section below for a detailed solution.

   (b) There are $\frac{n(n+1)(2n+1)}{6}$ regular squares and $\frac{(n-1)n^2(n+1)}{12}$ tilted squares for a total of $\frac{n(n+1)^2(n+2)}{12}$ grid squares. Again, see Presentation Suggestions for a detailed solution.

2. (a) Solution 1: Let’s label cells in a $7 \times 7$ square as shown below:

```
  1  2  3  4  5  6  7
 8  9 10 11 12 13 14
15 16 17 18 19 20 21
22 23 24 25 26 27 28
29 30 31 32 33 34 35
36 37 38 39 40 41 42
43 44 45 46 47 48 49
```

Now count the number of rectangles with the bottom right corner coinciding with the bottom right corner of cell 1; add the number of rectangles whose bottom right corner coincides with the bottom right corner of cell 2; add the number of rectangles whose bottom right corner coincides with the bottom right corner of cell 3; continue this process until we add the last number — that of the rectangles whose bottom right corner coincides with the bottom right corner of cell 49. We get the following sum:
\[
(1 + 2 + 3 + 4 + 5 + 6 + 7) + (2 + 4 + 6 + 8 + 10 + 12 + 14) \\
+ (3 + 6 + 9 + 12 + 15 + 18 + 21) + (4 + 8 + 12 + 16 + 20 + 24 + 28) \\
+ (5 + 10 + 15 + 20 + 25 + 30 + 35) + (6 + 12 + 18 + 24 + 30 + 36 + 42) \\
+ (7 + 14 + 21 + 28 + 35 + 42 + 49) \\
= (1 + 2 + 3 + 4 + 5 + 6 + 7) + 2(1 + 2 + 3 + 4 + 5 + 6 + 7) \\
+ 3(1 + 2 + 3 + 4 + 5 + 6 + 7) + 4(1 + 2 + 3 + 4 + 5 + 6 + 7) \\
+ 5(1 + 2 + 3 + 4 + 5 + 6 + 7) + 6(1 + 2 + 3 + 4 + 5 + 6 + 7) \\
+ 7(1 + 2 + 3 + 4 + 5 + 6 + 7) \\
= (1 + 2 + 3 + 4 + 5 + 6 + 7)(1 + 2 + 3 + 4 + 5 + 6 + 7) \\
= (1 + 2 + 3 + 4 + 5 + 6 + 7)^2 = \left(\frac{7 \cdot 8}{2}\right)^2 = 784
\]

Note: It might be helpful to start by looking at smaller cases — a 2×2 square, then a 3×3 square. In these two cases counting rectangles using the described method should present very little difficulty and will make the above computation clear. Let’s look at the 2×2 case:

<table>
<thead>
<tr>
<th>Cell no. X</th>
<th>All rectangles sharing their bottom right corner with cell no. X</th>
<th>Number of these rectangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1 1 2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3 1 3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4 3 4 2 1 2</td>
<td>4</td>
</tr>
</tbody>
</table>

Using this table we obtain the total number of all rectangles in a 2×2 square as follows:
\[(1 + 2) + (2 + 4) = (1 + 2) + 2(1 + 2) = (1 + 2)(1 + 2) = (1 + 2)^2\]

**Solution 2:** Every rectangle is defined by two vertical and two horizontal sides. Thus, to specify a rectangle, we need to choose 2 vertical lines out of 8 vertical lines in the \(7 \times 7\) square — this can be done \(\left(\begin{array}{c} 8 \\ 2 \end{array}\right)\) ways — then we need to choose 2 horizontal lines, which again can be done \(\left(\begin{array}{c} 8 \\ 2 \end{array}\right)\) ways. Hence the total number of rectangles is \(\left(\begin{array}{c} 8 \\ 2 \end{array}\right) \cdot \left(\begin{array}{c} 8 \\ 2 \end{array}\right) = 784\)

(b) As in part (a), to specify a rectangle we need to choose 2 vertical sides out of \((n+1)\) vertical grid lines in an \(n \times n\) square, then choose 2 horizontal sides out of \((n+1)\) horizontal grid lines. This can be done

\[
\left(\begin{array}{c} n + 1 \\ 2 \end{array}\right) \left(\begin{array}{c} n + 1 \\ 2 \end{array}\right) = \left(\begin{array}{c} n + 1 \\ 2 \end{array}\right)^2 = \left(\begin{array}{c} (n+1)n \\ 2 \end{array}\right)^2 = \frac{(n+1)^2 n^2}{4}
\]

ways.

Therefore, the number of such rectangles in an \(n \times n\) square is \(\frac{(n+1)^2 n^2}{4}\).

**Note:** Counting these rectangles by a different method we can express their number as

\[1^3 + 2^3 + 3^3 + \ldots + n^3\]

(See [1]). Thus we have proven that

\[1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{(n+1)n}{2}\right)^2 = (1 + 2 + 3 + \ldots + n)^2.\]

This last identity can be also illustrated as follows:
The next picture illustrates the case when $n = 5$:

These illustrations would also be helpful if you want to prove the identity rigorously and use induction on $n$, and notice that according to the illustration it might be a good idea to consider two cases, when
3. Let us draw a tilted square as shown below. Obviously, a grid line through a vertex crosses the opposite side at its midpoint. Thus the diagram represents the construction required in the problem.

Now it is clear that the figure inside is a square of area 1 while the tilted square has the area of $2^2 + 1^2 = 5$.

Hence its area is $\frac{1}{5}$ of the area of the original figure.

Now consider a few other tilted squares:

In the tilted square with $a = 3$ and $b = 1$, we have: $r = \frac{1}{3}$; the area
of the inner square is \((3 - 1)^2 = 4\), while the area of the tilted square is \(3^2 + 1^2 = 10\). Thus the ratio of the area is \(4/10\).

In the tilted square with \(a = 5\) and \(b = 2\), we have: \(r = \frac{2}{5}\); the area of the inner square is \((5 - 2)^2 = 9\), and the area of the tilted square is \(5^2 + 2^2 = 29\). Hence the ratio of the areas is \(9/29\).

In general, using the same idea of representing the given square by a tilted one we get the following result: if \(r = \frac{m}{n}\), where \(m < n\) and \(m\) and \(n\) are positive integers, the area of the inner square is \((n - m)^2\), the area of the outer square is \(m^2 + n^2\), and so the ratio of the areas is

\[
\frac{(n - m)^2}{n^2 + m^2} = \frac{(n - m)^2/n^2}{(n^2 + m^2)/n^2} = \frac{(1 - r)^2}{1 + r^2}.
\]

The formula remains true even when \(r\) is irrational.

**Presentation Suggestions**

The main goal of the warm-up problems is to familiarize students with the grid and start “respecting the grid” so that they are truly mindful of the grid lines, grid points and cells. You can easily skip these problems if you don’t have sufficient time.

It would normally take a 75–90 minute session just to do Problem 1. This problem is rich with ideas and can lead to deep and intriguing investigations in a variety of subjects such as combinatorics and number theory. Don’t rush — the time spent on truly understanding this problem is time well spent.

Start by posing Problem 1. Let students work on it for 5–10 minutes; make sure that they draw a \(7 \times 7\) square and start counting. After some students get an answer, have them come to the board and write their answers for all to see. There might be many different answers, such as 49, or 50, or even “infinite.” Hopefully, some students will have found the number to be 140. Discuss with them why some (or even all) of these answers are ‘correct’ — for example, if we only count the grid cells (i.e., \(1 \times 1\) squares) then we clearly have exactly 49 squares; if we totally disregard grid lines then there are, indeed, infinitely many squares which can be drawn in a \(7 \times 7\) square. Then have a student who found 140 squares explain what he or she did to find that number. Prompt them to list the number of squares according to their sizes, thus creating the following table on the board:
Students may notice something interesting: the number of squares of each size is a square number! But why? Let students think about it for a few minutes or until someone explains the phenomenon. If nobody comes up with a good explanation, show them the reason using a particular size — say, 3 × 3 squares. Suppose that you have a model of a 3 × 3 square with a colored dot at some place, for example, at the center of the upper left cell. Place this square at the upper left corner of the 7 × 7 grid. What are all other possible positions for a 3 × 3 square in the grid? To get all these positions systematically, start moving your square horizontally as far to the right as possible; make a mental note of the position of the colored dot after each move. When it is no longer possible to continue this process of sliding the square horizontally, return it to its leftmost position. Now slide it just one row down, and start moving it again to the right. Repeat these steps as long as possible, keeping in mind where the colored dot resides for each position of the 3 × 3 square. We get the following array of dots, and so there are twenty-five 3 × 3 squares in the grid.

<table>
<thead>
<tr>
<th>Square size</th>
<th># of these squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 × 1</td>
<td>49</td>
</tr>
<tr>
<td>2 × 2</td>
<td>36</td>
</tr>
<tr>
<td>3 × 3</td>
<td>25</td>
</tr>
<tr>
<td>4 × 4</td>
<td>16</td>
</tr>
<tr>
<td>5 × 5</td>
<td>9</td>
</tr>
<tr>
<td>6 × 6</td>
<td>4</td>
</tr>
<tr>
<td>7 × 7</td>
<td>1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>140</strong></td>
</tr>
</tbody>
</table>

Now we have a good explanation for the total number of 140 squares. But have we actually counted all squares? Are there any squares defined, in some way, by the grid that we have not accounted for as yet?
Ask students in what way did the grid 'define' all the squares we have counted so far. The answer is that their edges were on grid lines. For convenience, we will call these “regular squares.” But the grid has something else besides lines: grid points (the points of intersection of grid lines). Of course, the vertices of every regular square are grid points, but there are squares whose vertices are grid points but whose edges do not lie on grid lines. We will call these “tilted squares.” Challenge the students to draw a number of different tilted squares; give them time to discover a fair amount of different orientations and sizes. Below are just three possible examples.

So now we will try to find the number of all tilted squares in a $7 \times 7$ grid. Is there any good systematic way to count them?

Students might come up with several suggestions, such as (a) the side length; (b) the slope of one side; (c) the size. At this point, it will be a very natural place to discuss what is it that we call the size of a square, or more generally, the size of a flat figure. You should wait until someone says that it is the area (if need be, lead them to this conclusion by asking, “What do we mean by the size of a piece of straight line? A piece of a curve?” To be sure, it is the length: if we know that one piece of a curve has the length of 2 units, and another has the length of 3 units, it tells us that the latter piece is of a bigger size).

Thus we can systematize our counting in a way similar to that used for counting regular squares — by listing possible areas (sizes) of tilted squares, then finding out what is the total number of squares of each area. This seems to be a good plan. Are there any problems with this approach? Let students think for a while; then ask what is the area of the smallest tilted square? It might help if they find the areas of all three tilted squares pictured above. Let students work
trying to find the answer for 10–15 minutes. If nobody comes up with a general method, give them a hint: without saying anything, just outline the smallest regular square containing the tilted one, as shown below:

![Diagram of a tilted square and its circumscribed regular square](image)

They should notice that the circumscribed regular square has an area of 9, and to get the area of the tilted square we need to subtract the total area of four triangles that lie outside of the tilted square. Each of these four triangles is exactly half of a 2 by 1 rectangle; if we pair them up, we see that the total area is that of two such rectangles. Therefore, the area of the tilted square is

\[
3^2 - 2 \cdot (2 \cdot 1) = 9 - 4 = 5.
\]

Similarly, the area of the smallest tilted square is

\[
2^2 - 2 \cdot (1 \cdot 1) = 4 - 2 = 2,
\]

and the area of the third tilted square shown above is

\[
5^2 - 2 \cdot (3 \cdot 2) = 25 - 12 = 13.
\]

Now we can generalize the above observation as follows. Let us consider any tilted square. Start with its lowest vertex, and go to the next one (counterclockwise). Since it is tilted, we should go \(a\) cells to the right and \(b\) cells up. To get to the next vertex, we must go \(a\) cells up and \(b\) cells to the left (check that students see the necessity of this — otherwise, the edges will not be of the same size and perpendicular to each other (why?)). Continuing in this manner, we get the following picture:
Thus the area of the tilted square is

$$(a + b)^2 - 2 \cdot (a \cdot b) = a^2 + 2ab + b^2 - 2ab = a^2 + b^2.$$ 

Note: If we denote the length of an edge of the tilted square by $c$, then obviously its area is $c^2$. So from the above computation, we see that $c^2 = a^2 + b^2$. So we just proved Pythagorean Theorem! (in the special case when the length of each side of a right triangle is an integer)

We conclude that a positive integer is the area of a tilted square if and only if it can be written in the form $a^2 + b^2$, where $a$ and $b$ are positive integers. Ask students to check which of the consecutive integers starting with 1 can be written in this form. Give them a few minutes for the task; they should get the following possible areas:

$$
2 = 1^2 + 1^2 \\
5 = 1^2 + 2^2 \\
8 = 2^2 + 2^2 \\
10 = 1^2 + 3^2 \\
13 = 2^2 + 3^2 \\
17 = 1^2 + 4^2
$$

Let us give regular and tilted squares a collective name of ‘grid squares.’ We can state that a positive integer is the area of a grid square if and only if it can
be written in the form \( a^2 + b^2 \), where \( a \) and \( b \) are integers, \( a \) is positive, and \( b \) is either positive (for a tilted square) or 0 (for a regular square).

We summarize our observations as follows:

- The following numbers represent areas of grid squares: 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, …

- The following numbers can’t be areas of grid squares: 3, 6, 7, 11, 12, 14, 15, …

A profound question is, “what characterizes each of these two groups of numbers?” You should challenge your students to investigate this question (even though it leads deep into number theory, students could extend the lists and then try to make conjectures of their own; using computers might help to get large lists thus providing them with more data for conjectures).

Returning to the task of counting the tilted squares in a \( 7 \times 7 \) grid, we know that every tilted square can be inscribed in a regular square, so we should look at regular squares and see how many tilted squares can be inscribed in each of them.

Each \( 2 \times 2 \) regular square contains only 1 tilted square.

Each \( 3 \times 3 \) square contains two different tilted squares, since \( 3 = 1 + 2 = 2 + 1 \) (the picture below illustrates this):

Likewise, each \( 4 \times 4 \) square contains three different tilted squares, since \( 4 = 1 + 3 = 2 + 2 = 3 + 1 \). (Have students draw the corresponding pictures.)

Now we can add two additional columns to our first table:
<table>
<thead>
<tr>
<th>Grid square size</th>
<th># of grid squares</th>
<th># of tilted squares per grid square</th>
<th>Total # of tilted squares inscribed in regular squares of this size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 1$</td>
<td>49</td>
<td>0</td>
<td>$49 \cdot 0 = 0$</td>
</tr>
<tr>
<td>$2 \times 2$</td>
<td>36</td>
<td>1</td>
<td>$36 \cdot 1 = 36$</td>
</tr>
<tr>
<td>$3 \times 3$</td>
<td>25</td>
<td>2</td>
<td>$25 \cdot 2 = 50$</td>
</tr>
<tr>
<td>$4 \times 4$</td>
<td>16</td>
<td>3</td>
<td>$16 \cdot 3 = 48$</td>
</tr>
<tr>
<td>$5 \times 5$</td>
<td>9</td>
<td>4</td>
<td>$9 \cdot 4 = 36$</td>
</tr>
<tr>
<td>$6 \times 6$</td>
<td>4</td>
<td>5</td>
<td>$4 \cdot 5 = 20$</td>
</tr>
<tr>
<td>$7 \times 7$</td>
<td>1</td>
<td>6</td>
<td>$1 \cdot 6 = 6$</td>
</tr>
<tr>
<td><strong>Total:</strong></td>
<td><strong>140</strong></td>
<td></td>
<td><strong>Total:</strong> <strong>196</strong></td>
</tr>
</tbody>
</table>

Hence the total number of all grid squares in a $7 \times 7$ square is $140 + 196 = 336$.

If students are ready to do some algebraic computations we are now in a position to generalize our previous work.

The total number of regular squares in an $n \times n$ square is

$$n^2 + (n-1)^2 + (n-2)^2 + \cdots + 2^2 + 1^2 = \frac{n(n+1)(2n+1)}{6}$$

The total number of tilted squares in an $n \times n$ square is

$$(n-1)^2 \cdot 1 + (n-2)^2 \cdot 2 + (n-3)^2 \cdot 3 + \cdots + (n-(n-1))^2 \cdot (n-1)$$

$$= (n^2 - 2n \cdot 1 + 1^2) \cdot 1 + (n^2 - 2n \cdot 2 + 2^2) \cdot 2 + (n^2 - 2n \cdot 3 + 3^2) \cdot 3 + \cdots$$

$$+ (n^2 - 2n \cdot (n-1) + (n-1)^2) \cdot (n-1)$$

$$= n^2(1 + 2 + 3 + \cdots (n-1)) - 2n(1^2 + 2^2 + 3^2 + \cdots + (n-1)^2) + (1^3 + 2^3 + 3^3 + \cdots$$

$$+ (n-1)^3)$$

$$= \frac{n^2(n-1)n}{2} - \frac{2n(n-1)n(2n-1)}{6} + \frac{(n-1)^2n^2}{4} = \frac{(n-1)n^2(n+1)}{12}$$

Finally, the total number of all grid squares in an $n \times n$ grid is

$$\frac{n(n+1)(2n+1)}{6} + \frac{(n-1)n^2(n+1)}{12} = \frac{n(n+1)^2(n+2)}{12}$$

Comparing the last two formulas, we see that the number of all grid squares in an $n \times n$ grid is exactly the same as the number of tilted squares in an $(n+1) \times (n+1)$ grid. But why? Can we find a ‘natural’ one-to-one correspondence
between these two sets? Pose this question to your students and let them brood over it.

An interesting and surprising use of tilted squares can be seen in solving Problem 3. If instead of drawing a given square as a regular one we tilt it a solution to the problem becomes obvious. Every statement we made in the presented solution can be justified using some elementary geometry; if your students are interested and ready you might ask them to identify and prove every statement in the official solution which needs justification.

References
