Family Circle: Chessboard and Tilings Mathematics

**INTRO** This time we had a good mixture of students, teachers, parents/guardians, and math friends. All students came from two schools – Tuba City Boarding School, and Hopi Junior/Senior High School. Bluebird congratulates these students and their teachers! We also had people from seven different states (Arizona, California, Connecticut, Michigan, New Mexico, New York, North Carolina), and even from Alberta, Canada.

As usual, we started this meeting by listening to a bluebird song and thinking of whether we had any questions for Bluebird. Some people did! Those questions are listed at the end of this recap.

Next we looked together at the Issue 4. We started by reading a quote by Jerry C. Elliott-High Eagle. He shared his thoughts about two main obstacles in the way of learning and discovery – the fear of letting go of what is known and comfortable, and the fear of stepping into unknown and uncharted territory. We are grateful to Jerry, a great Native American, a good friend and a valuable advisor of AIMC, for sharing his wisdom. Take a look at his amazing biography sketch at https://aimathcircles.org/our-people/ (scroll to the Advisory Board).

Then we had a short discussion about two problems below:

**Problem 1:** A rectangular cornfield is painted black and white in a checkerboard pattern, so that it looks like a 100-by-85 chessboard. Tawa (Sun) is standing with a large sack of corn kernels in one corner of the field, and at this point, the field is empty. He can take a step from any one of the squares to one of the adjacent squares (up, down, left, or right, but not diagonally). When he moves to a new empty square, he puts a kernel down on it. But if the new square already has a kernel, Tawa removes the kernel.

Is it possible for Tawa to walk around the field in such a way that in the end there is exactly one kernel lying in each of the black squares and no kernels lying in the white squares?

**Problem 2:** Count the number of possible ways to tile the following shapes with 2-by-1 dominos. How many ways to tile shape #1? #2? #3? Can you guess what the next shape would be like, and how many ways are there to tile it?

These shapes can be found in much Indigenous art. The motif is shared between many nations. Do you recognize it?

At the circle, we used Desmos to draw together:

https://teacher.desmos.com/activitybuilder/custom/60995a01606058068476f4ef

- To play as a math circle participant, follow the link, then click the Student Preview button.
- To remake this activity for your group as a teacher, click (three dots) at the top right corner, and select Copy and edit from the drop-down menu.
We agreed that in Problem 1 the field is way too big, so it would make sense to start with a smaller case, and we were able to solve the problem with a 2-by-2 field.

In Problem 2, we tiled shape 1 together, and we’ve agreed that there were exactly two ways to tile it.

And at this point we went to four breakout rooms. In rooms 1 and 2 people tackled Problem 1, and people in Rooms 3 and 4 worked on Problem 2.

After 30 minutes everyone came back together again to share their discoveries with the entire group (even though someone complained loudly, “Oh, we’re going away [from the breakout room] just as we’re solving the problem.”). Well, if you like a problem, just keep thinking about it, and you will find more and more interesting things. It might take years until you are completely satisfied.

Bluebird is grateful to three fearless students – Quaidin B., Ciera T., and Ye-Shiao T. – for reporting their teams’ results. They did a marvelous job!

**Solving Problem 1**, people found many different ways to achieve the goal for small size fields. They asked some additional questions, for example, what is the least number of moves it takes to have a kernel on every black square and no kernels on white squares for a field of a given size?

But very importantly, they realized that one needs to find a *general strategy* that would work for a field of any size. One such strategy is illustrated below using a 2-by-5 field. See if you can adopt it for larger fields.

We start by numbering the squares. Our path will lead through squares by their numbers ($1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots$) so that we don’t miss any square.

Now we are going to show each move; a dot means a kernel, and a red boundary indicates the square where Tawa is at the end of a move.

In this notation, a “+” on top of a number means that the square with that number has a kernel (and a “−” would mean no kernel in that square).

The whole series of the next moves which results in getting what we want is here:

\[
\begin{align*}
\frac{+}{2} & \rightarrow \frac{+}{3} \rightarrow \frac{-}{2} \rightarrow \frac{-}{3} \rightarrow \frac{+}{4} \rightarrow \frac{+}{5} \rightarrow \frac{-}{6} \rightarrow \frac{+}{7} \rightarrow \frac{-}{8} \rightarrow \frac{+}{7} \rightarrow \frac{-}{8} \rightarrow \frac{+}{9}
\end{align*}
\]
We can also show all these moves pictorially:

There are many different ways to capture moves. Try and invent your own! Here is how the screens looked when participants discussed their moves:

Now let’s go to Problem 2. One of the questions in the problem is to figure out what the next shape in the sequence looks like. Both teams working on this problem came up with an answer, but their answers were different!

In room 3, people decided to use the idea of the inner square. For example, shape two has the inner square of side 2, and shape 3 has the inner square of side 4. So they concluded that all subsequent shapes will have inner squares with sides 6, 8, 10, etc., and then each side of the inner square will have additional pieces like “step pyramids.”
Thus the next two shapes would look like this:

![Shapes](image)

They also calculated the total number of grid squares in each of their shapes:

Shape one: \(2 \times 2 = 4\) (it’s just a 2-by-2 square)

Shape two: \(2 \times 2 + 4 \times 2 = 12\) (it’s a 2-by-2 square plus 4 ‘ears’ with 2 squares each)

Shape three: \(4 \times 4 + 4 \times 2 = 24\) (it’s a 4-by-4 square plus 4 ears with 2 squares each)

Shape four: \(6 \times 6 + 4 \times (4 + 2) = 60\) (can you see why we calculate it like this?)

Shape five: \(8 \times 8 + 4 \times (6 + 4 + 2) = 112\) (can you see how we got the number?)

The other team (room 4) discerned a different pattern: If we start with a given shape, we can add one more grid square above, below, to the left, or to the right of each boundary grid square (but not diagonally). For example, to go from shape one to shape two we add the squares shown in red:

![Red squares](image)

Similarly, to go from shape two to shape three we add the squares shown in black:

![Black squares](image)
Continuing in the same fashion, we get the next two shapes:

![Shapes](image)

Notice that the right shape here is exactly like shape number 4 in the sequence obtained by the team in room 3! But the left shape in the above diagram didn’t appear in room 3 sequence at all.

Question: Can you continue each sequence? What do you notice?

Let’s go back to the sequence of shapes obtained by the room 4 team. They also calculated the number of grid squares in each shape, and their numbers were 4, 12, 24, 40, 60. They noticed that

\[
\begin{align*}
4 &= 4 \times 1 \\
12 &= 4 \times 3 = 4 \times (1 + 2) \\
24 &= 4 \times 6 = 4 \times (1 + 2 + 3) \\
40 &= 4 \times 10 = 4 \times (1 + 2 + 3 + 4) \\
60 &= 4 \times 15 = 4 \times (1 + 2 + 3 + 4 + 5)
\end{align*}
\]

This observation combined with noticing a 4-fold symmetry of the shapes led them to a new way of constructing the shapes: you start with a ‘staircase’ of the desired size, then use the symmetry to complete the shape. Below the initial ‘staircase’ is shown in red while it’s 3 symmetric copies are shown in black:

![Staircase and copies](image)
There was not much time left, so it is still a challenge to count the number of domino tilings for each shape. If you continue to play with this problem, let Bluebird know what you’ve found.

Share your ideas with other Bluebird Math Circle participants at https://aimathcircles.org/Bluebird

Bluebird also wants to share with you a few images of indigenous art where you can see designs used in Issue 4 problems:

Hopi pottery by Ethel Youvella

Hopi pottery by Dee Setalla

Navajo rug

Hopi Pottery by Colleen Poleahla

Here’s a problem inspired by this image:

How many ways are there to tile the checkered part of the design with 2-by-1 dominos? Same question but if you remove the 3 squares on the top left. Of course we assume that we put the design on a flat surface and all squares are of the same size.
Finally, we'd like to quote one of the participants who said:

“I was kind of scared with the checkerboard and I thought it was gonna be very challenging for me, but we had a good team and we worked it out.”

So this is the team's power in action!

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**New Questions for Bluebird**

*What is a cube root?* – from Ye-Shiao T.

*What is the Bluebird’s favorite mathematical equation?* – from Mary Clarke

*What is the difference between geometry, mathematics, trigonometry, calculus, and precalculus?* – from Ciera T.

**BLUEBIRD SAYS**—Curious questions. I will fly around and seek some answers. Watch this space in the next flyer!

Submit your math-related questions at [https://aimathcircles.org/Bluebird](https://aimathcircles.org/Bluebird)