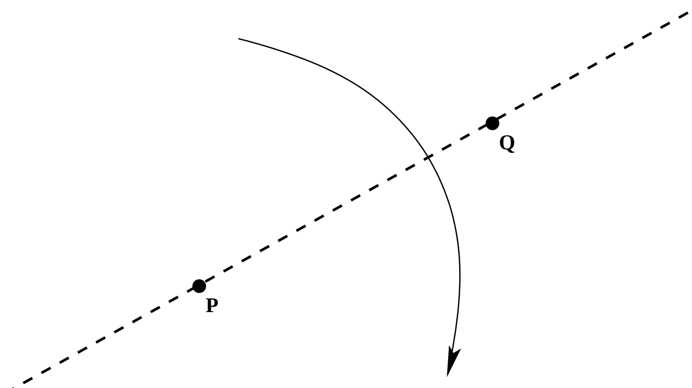
Origami Axioms

Given a piece of paper, it is possible to fold lots of different lines on it. However, only some of those lines are *constructible* lines, meaning that we can give precise rules for folding them without using a ruler or other tool. Each fundamental folding rule is called an origami *axiom*.

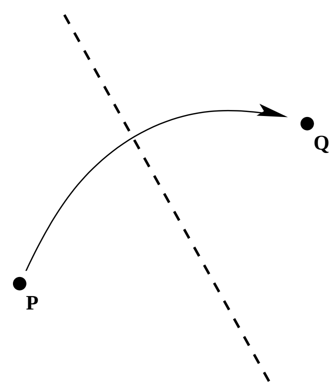
When we start with a square piece of paper, we begin with four marked lines (the four edges) and four marked points (the four corners). Any crease created by applying an origami axiom to existing marked points and lines is a new marked line. Any place where two marked lines cross is a new marked point.

There are seven origami axioms in all.

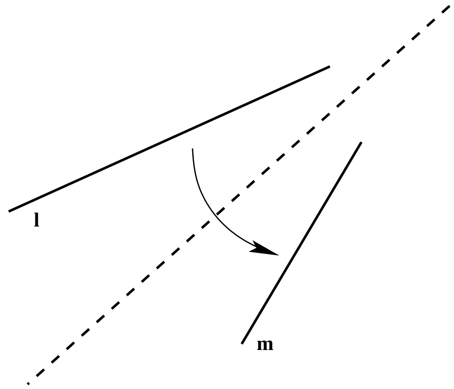
- O1 – Given two marked points, we can fold a marked line connecting them.



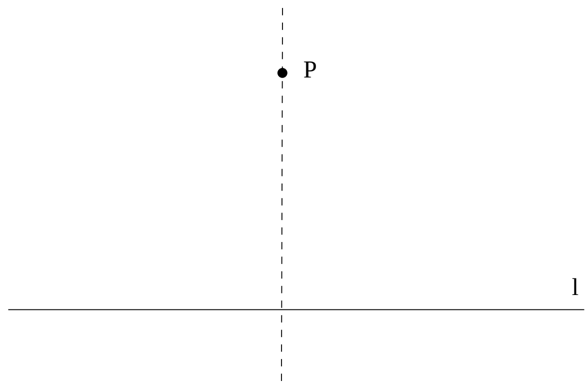
- O2 – Given two marked points P and Q, we can fold a marked line that places P on top of Q.



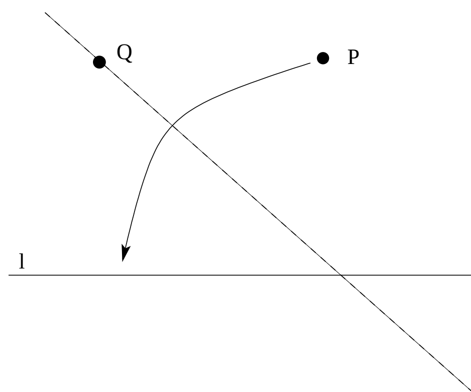
- O3 – Given two marked lines l and m, we can fold a marked line that places l on top of m.



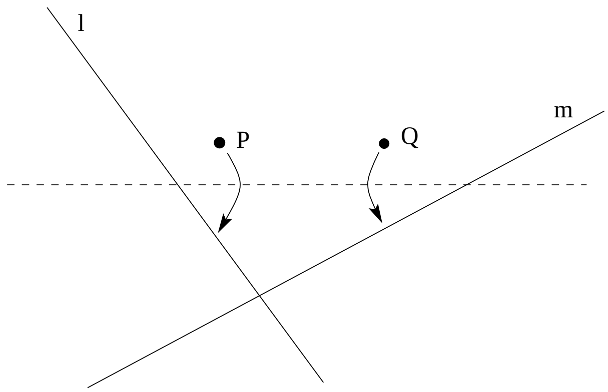
- O4 – Given a marked point P and a marked line l, we can fold a marked line perpendicular to l passing through P.



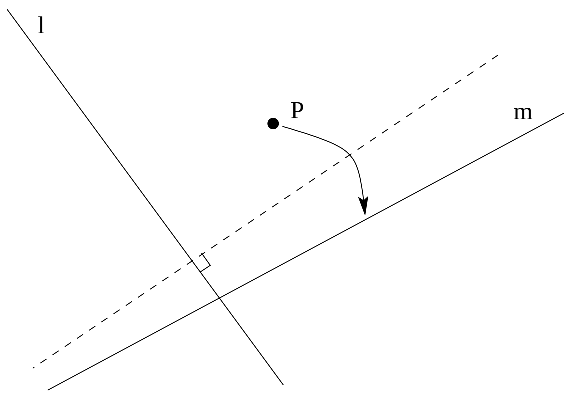
- O5 – Given two marked points P and Q and a marked line l, we can fold a marked line passing through Q that places P on l.



- O6 – Given two marked points P and Q and two marked lines l and m, we can fold a marked line that places P on l and Q on m.



- O7 – Given a marked point P and two marked lines l and m, we can fold a marked line perpendicular to l that places P on m.



Restrictions on applying these axioms

- O1 – The fold exists and is unique for any two distinct points.
- O2 – The fold exists and is unique for any two distinct points.
- O3 – The fold exists and is unique for any two distinct lines.
- O4 – The fold exists and is unique for any point and any line.
- O5 – The fold does not always exist, and there can be up to two different folds that satisfy it. In this axiom, the point P is the focus for a parabola and the line l is its directrix. The assertion is that we can find a tangent line for the parabola through Q. There are no tangent lines through points in the interior of the parabola. Therefore, if Q lies inside the parabola determined by P and l, no tangent fold exists. If Q is any point outside of the parabola, two tangent folds exist. If Q is on the boundary, there is exactly one tangent fold. If P lies on l, the parabola is infinitely skinny and has no interior. Thus, in this case, Q can be anything and this axiom becomes equivalent to O4.
- O6 – The fold does not always exist and it is not unique in general. In this axiom, P is the focus for a parabola with directrix l and Q is the focus for a parabola with directrix m. Since folding a point to a line always gives us a tangent to the parabola they determine, the action of taking two points to

two lines makes the fold a tangent for both parabolas simultaneously. There are at most three such tangents for two parabolas, making this problem a cubic in general. There can also be two tangents, one tangent, or no tangents.

- O7 – This axiom is equivalent to O6 in the case where one of the points is on one of the lines.

O1 through O5 are sufficient to duplicate any straightedge and compass construction

Here is how constructibility works. We are given the points $(0, 0)$ and $(1, 0)$ in the plane. We want to know which points in the plane can be constructed by straight-edge and compass. These constructible points are precisely those with coordinates which are solutions to some equation $ax^2 + bx + c = 0$, where a, b, and c are integers. Using the quadratic formula, we know that the solutions of this equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where a, b, and c are integers.

Thus, we need to verify that we can use the origami axioms to add, subtract, multiply, and divide given lengths. We also need to be able to take the square root of a given length.

Adding and subtracting lengths

To add two given lengths, we need to be able to copy a length from one line segment to a particular place on another. One way to do this is to use O3 to fold the first line onto the second. This will place the segment somewhere on the line. We now need to move one end of the segment to a particular point, and make the other end of the segment lie in our preferred direction on the line. It is possible that the point we are trying to hit lies in the middle of the segment. In this case, we first use O4 to make a perpendicular fold through one of the end points, we copy the segment to that part of the line, and then unfold. Now we have a segment which does not touch the point we want to hit. We use O2 to fold the near endpoint of the segment to the target point. The segment may or may not be going in the desired direction. If it is not, we use O4 through the target point to fold the line segment in the other direction.

Notice that by copying one segment to the end point of another segment so that they both lie on the same line will allow us to add the lengths. To subtract lengths, we need to copy the segment on top of the other one to find the difference in their lengths.

Multiplication and division

To multiply two given segments of length a and b, we first place them so that they form an acute angle. We can do this using the copying lengths methods already discussed. Next, we copy the unit length segment onto the line containing segment b so that one end of the unit length segment lies at the angle vertex.

We now use O1 to create a line from the end of a to the end of the unit segment. We now need to construct a parallel line through the point at the end of b. We can use O4 twice to do this, for example. Now we mark the point on the line containing a which intersects this parallel line. The length from the vertex to this point is ab by similar triangles.

We use a similar procedure to divide a by b. The set-up is the same, but this time, we use O1 to connect the end of a to the end of b. Now we construct a parallel line through the end of the unit length. The point on the line containing a which intersects this parallel is the end point of a segment of length a/b.

Finding a square root

A good way to take the square root of a length n is to ask the parabola $y = x^2$ to do it for you. We begin by copying n onto the y-axis. We construct the horizontal line $y = n$ at this height using O4 twice.

Next we mark the focus at $(0, 1/4)$ and a point at $(0, -n)$ (which are both constructible). We then use O5 to create a fold through the focus which takes the other endpoint to the horizontal line. We know the image point will be on the parabola $y = x^2$ because the distance from the focus to the image point is equal to $n + 1/4$, which is also the distance from the image point to the directrix at $y = -1/4$. The two points on the horizontal line where the image point can be are therefore $\sqrt{n}$ units away from the vertical line $x = 0$.

We know that it is not always possible to use O5, so let us consider whether we have used it safely in this construction. When we use O5, we are using the horizontal line at $y = n$ as the directrix of the parabola whose tangent line we are constructing. We are using $(O, -n)$ as the focus of this parabola. The parabola therefore has vertex at (0,0) and opens downwards. Since the point we are finding a tangent through is at $(0, 1/4)$ the desired tangent line exists and so the construction is always possible.

O2, O5, and O6 are the only essential axioms

We used all five of the origami axioms in the constructions above, but we only really need O2 and O5 to accomplish O1, O3, and O4. We have not analyzed the power axiom O6 which allows us to trisect angles, double cubes, and otherwise solve cube roots. However, O7 is really O6 in disguise, as we pointed out earlier.

O4 is really a special case of O5 as we pointed out before.

O1 can be replaced by O2 and O5 together. First use O2 to construct the perpendicular bisector. Use O5 on the two original points and the constructed line to mark two points on the perpendicular bisector. Now use O2 to bring these points together. This constructs the line needed for O1.

O3 is also easily replaced by O2 and O5. If the two lines are parallel, we use the O4 version of O5 to construct a perpendicular to both of them and then use O2 to bring one intersection point to the other. If the two lines are not parallel, we first mark the intersection point and a different arbitrary point P on one of the lines. We use O5 to make a fold through the intersection point that brings point P onto the other line. This accomplishes O3.

O6 relates to a cubic equation

See Thomas Hull's article "Solving Cubics with Creases: The Work of Beloch and Lill" in the MAA monthly, April 2011.